

Introduction to Quantum Information- III



238 A GIRL OF THE LIME...
clothing lining one wall. She p...
armload and carried it out to...
piece she pitched into the green...
all those articles she had dur...
months from for years, and...
sucked them down. She...
gathered every scrap th...
Robert Comstock, exc...
threw it into the sw...
set her door wide open...
She was too weary...
rest drove her. Sh...
she never could w...
her judgment. At...
Mrs. Comstock cl...
around Elnora's r...
pictures were strang...
it with packages an...
with cocoons; moth...
where. Under the b...
white boxes. She did...
She pulled out one and...
covered with a sheet of...
ing in it were dozens of...
Each one was labelled, always...
in many cases four, showing unde...
both male and female. They were o...
shape.
Mrs. Comstock caught her breath sharply.

MARGARET REVEALS A...
here had Elnora gotten all of the...
quisite sight the woman...
the boxes to feast on their beautiful...
so there came more fully a sense...
her and her child. She could no...
had gone to school, and performed...
when it was finished, up to the...
other who should have been...
had been the one to bring...
Elnora had come to hate...
and replaced the boxes...
the room. This time...
she did not remember...
up one and found that...
iced over the first pages...
When the text reached...
laid it down. Then she...
ery chapters. By that time...
ded them. By that time...
jumble of ideas about cap...
baits and bright lights. Being unable...
thinking deeply. Being unable...
nothing else to do she glanced at...
preparing supper. The work dragged...
snatched up and dressed hurriedly. A...
as made in short order. Strawberries that...
intended for preserves went into shortcake. A...
delicious odours crept from the cabin. She put many

General Ideas

The Goal: Define and Calculate the following:

Classical Channel

Quantum Channel

One Capacity

Four different capacities

یک کد گذاری ساده برای تصحیح خطا

x $P(y|x)$ y

0 \longrightarrow 000

1 \longrightarrow 111

نرخ مخابره

$$R = \frac{1}{3}$$

This is a very simple **letter** code.

Block Codes

010010010010001001001111010010100101001

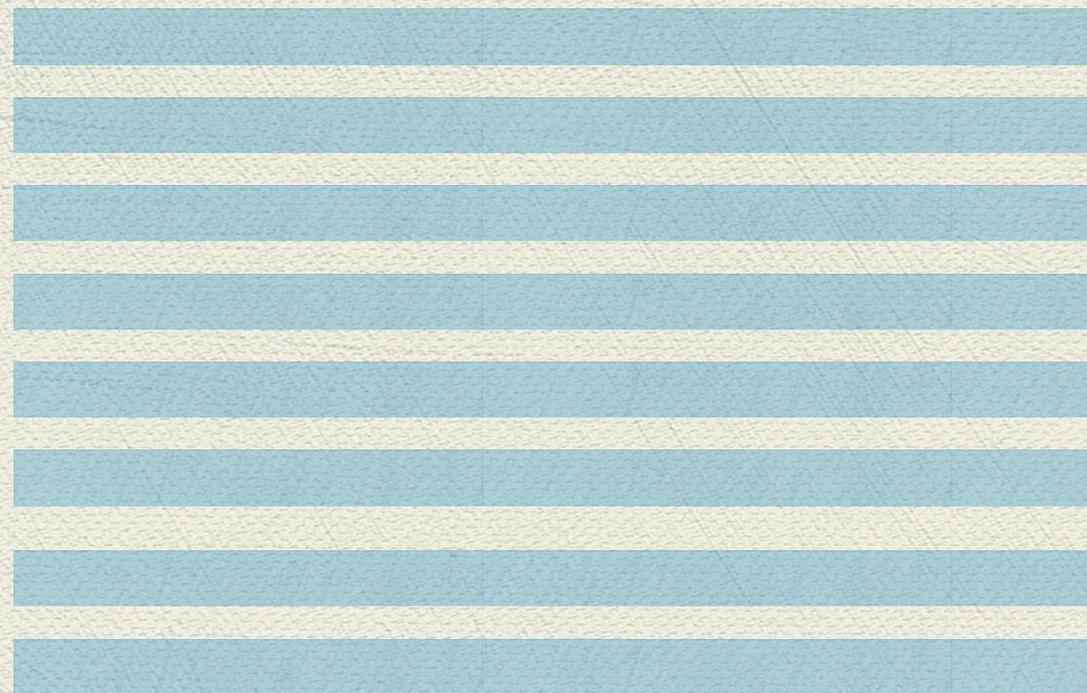
k

010010010001011011011000100100111101000110101101110100101001

n

$M_k \longrightarrow X^{(n)}$

Encoding



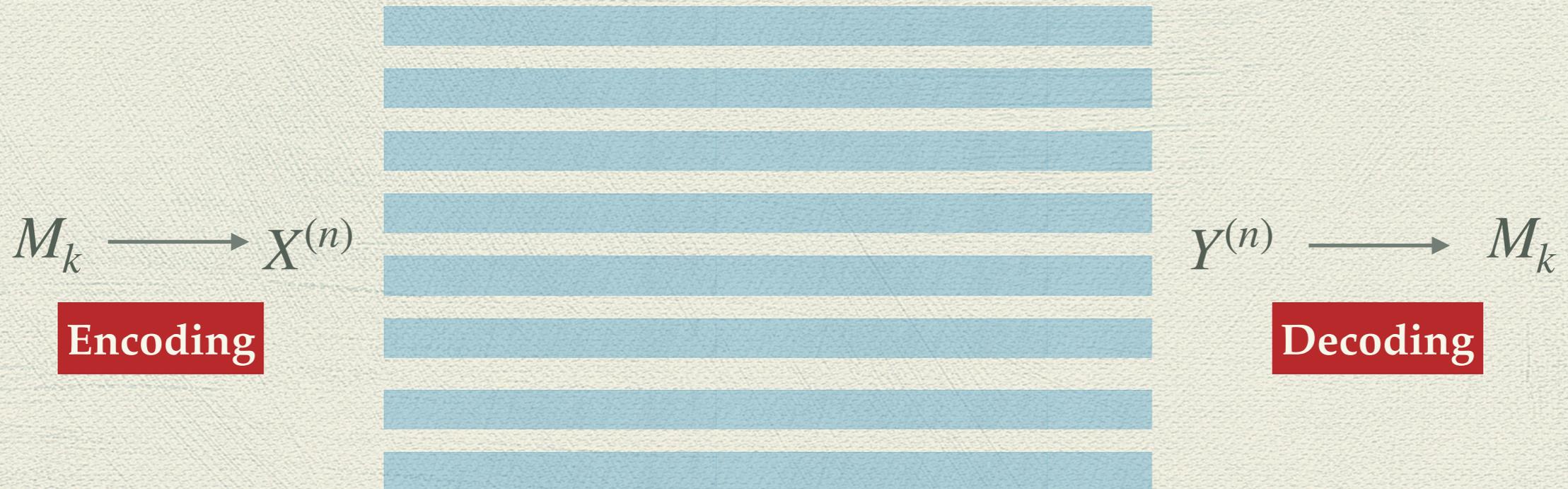
$Y^{(n)} \longrightarrow M_k$

Decoding

$$R(X) = \lim_{n \rightarrow \infty} \frac{k}{n}$$

$$P_{error} \longrightarrow 0$$

تعریف: ظرفیت کانال کلاسیک



$$C = \underset{X}{\text{Max}} R(X)$$

می بایست روی تمام منابع ها بیشینه نرخ را حساب کرد.

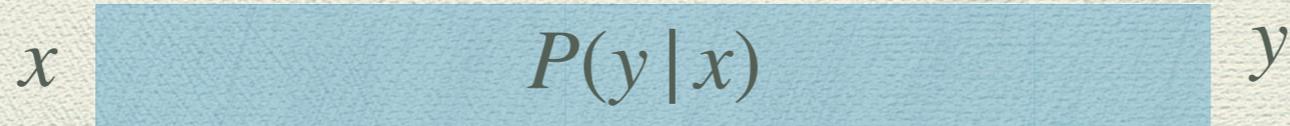
محاسبه: ظرفیت کانال کلاسیک

$$x \quad P(y|x) \quad y$$

$$I(X : Y) = H(X) + H(Y) - H(X, Y)$$

$$C = \underset{P(x)}{\text{Max}} I(X : Y)$$

محاسبه: ظرفیت کانال کلاسیک

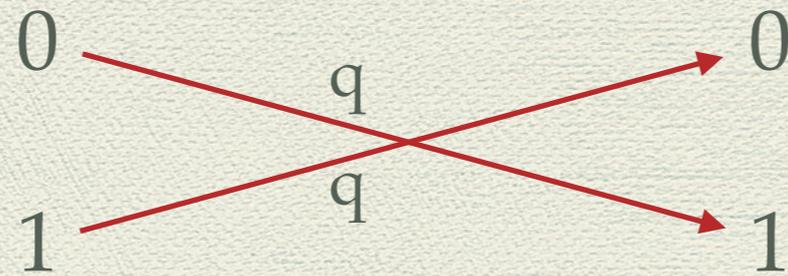


$$P(x, y) = P(y | x)P(x)$$

$$H(X) = - \sum_x P(x) \log P(x) \quad H(Y) = - \sum_y P(y) \log P(y)$$

$$H(X, Y) = - \sum_{x,y} P(x, y) \log P(x, y)$$

مثال: محاسبه ظرفیت یک کانال متقارن



$$I(X : Y) = q \log q + (1 - q) \log(1 - q) + [H(\lambda)]$$

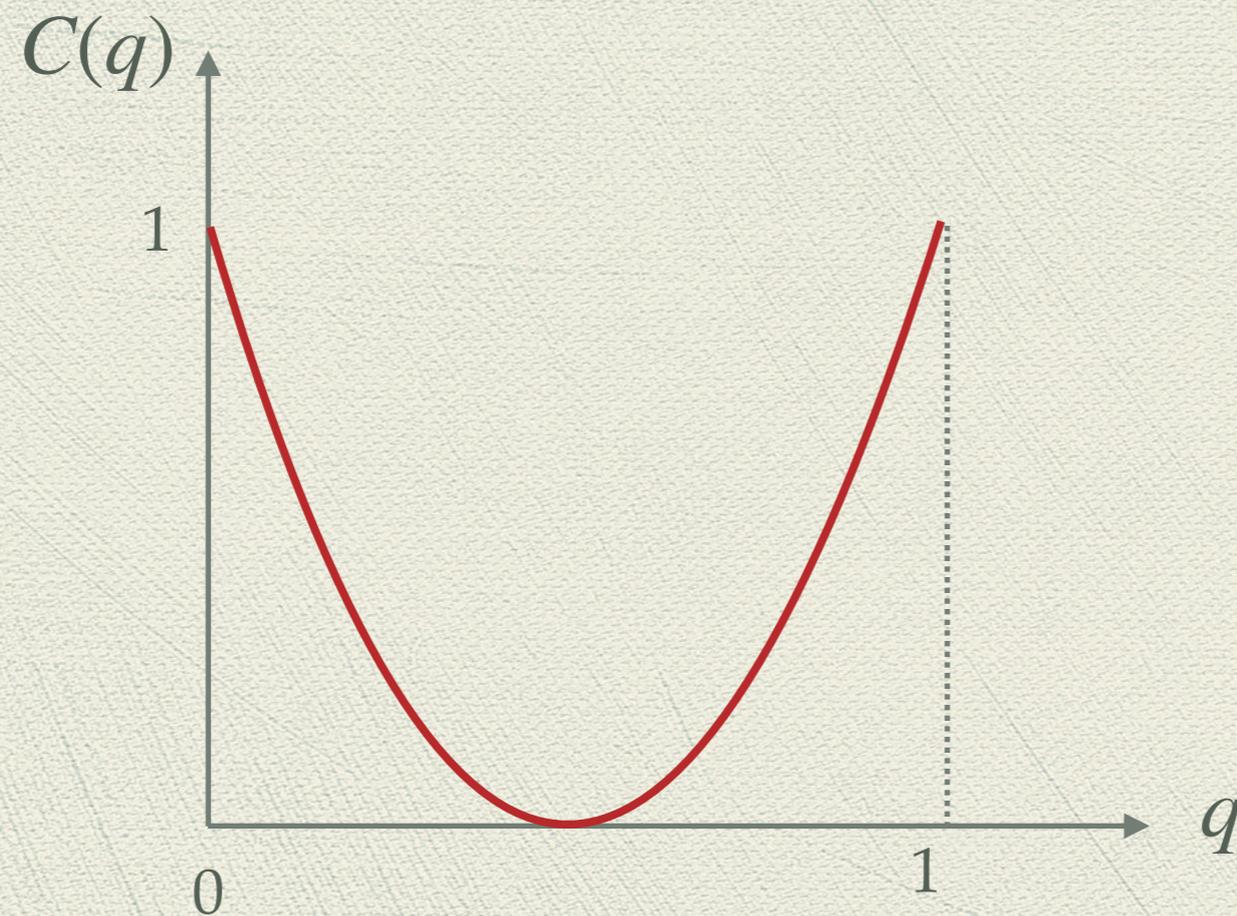
$$\lambda = p + q - 2pq$$

$$p + q - 2pq = \frac{1}{2} \longrightarrow p = \frac{1}{2}$$

بیشینه کردن

نتیجه نهایی محاسبه

$$C(q) = 1 + q \log q + (1 - q) \log(1 - q)$$



ظرفیت های کانال های کوانتومی



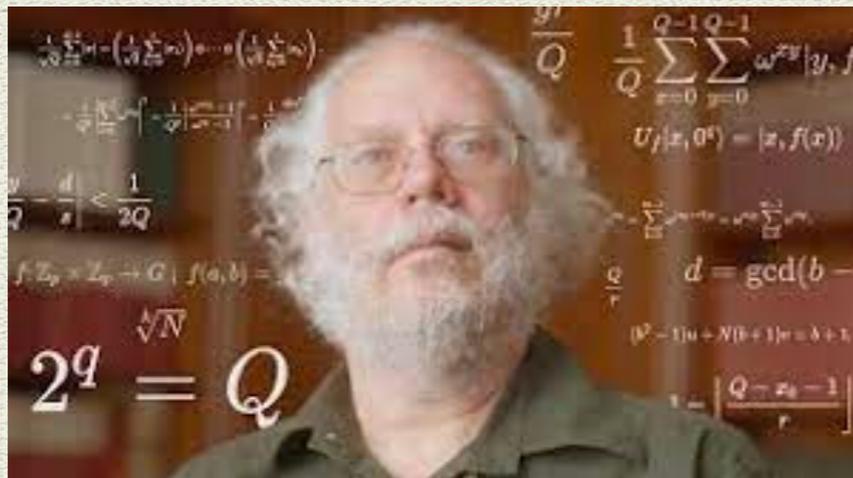
Holevo



Shumacher



Westmoreland



Peter Shor



Charles Bennett

یک- برای یک کانال کوانتومی انواعی از
ظرفیت تعریف می شود.

دو- تعریف این ظرفیت ها یک چیز است و تقلیل
این تعریف ها به یک رابطه بسته یک چیز دیگر.

سه -حتی محاسبه آن رابطه بسته هم آسان نیست
(به دلیل بیشینه کردن روی یک فضای پیوسته)

چهار- برخلاف ظرفیت کلاسیک این ظرفیت ها
عموما جمع پذیر نیستند (به دلیل درهم تنیدگی).

Q1-Classical Capacity of Quantum Channels

... have served to increase the perceived value of the reward and devalue the effort. What tends to happen in the English Cup in Russia, his attitude was calmer

Definition: Classical Capacity of Quantum Channels

M=Classical message of k-bits

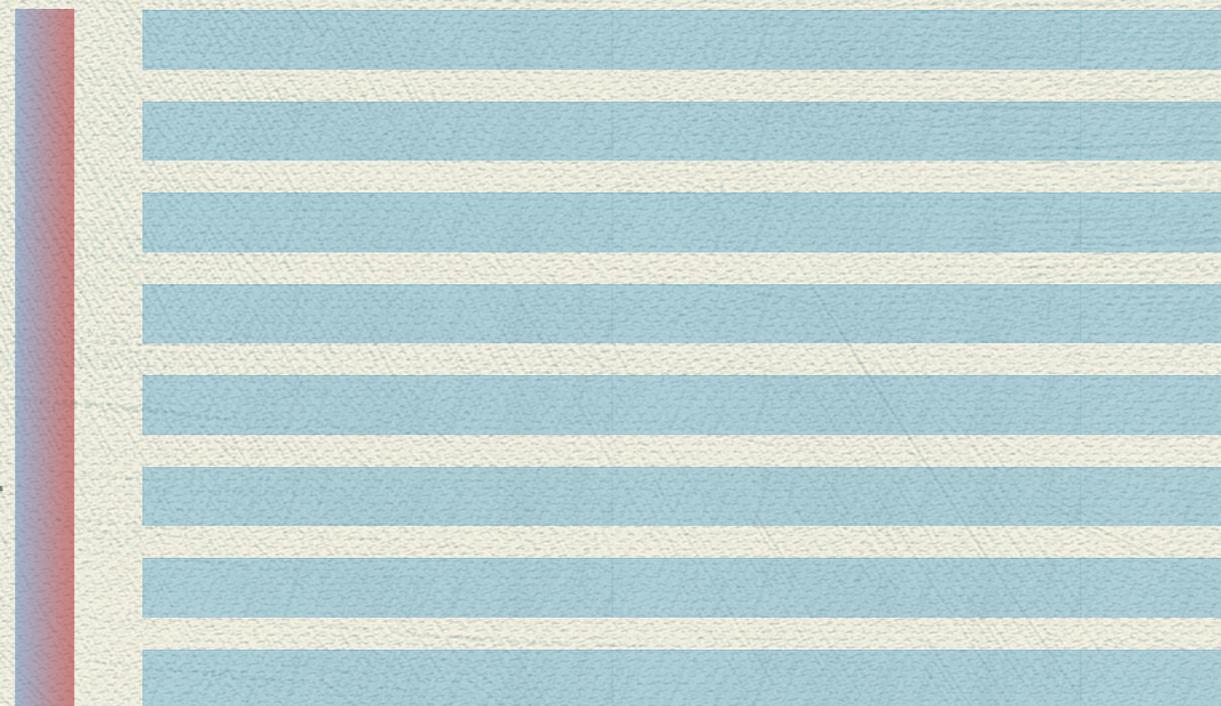


The states can be entangled.

Measurements can be entangled.

$|\psi^n(M)\rangle$ = Quantum state of n-qubits

$|\psi^n(M)\rangle$



$\{E_1, E_2, \dots\}$

این حالت ها بر هم عمودند.

این حالت ها ممکن است بر هم عمود نباشند
و عموماً هم آمیخته هستند.

Example:



$$00 \longrightarrow |\phi^+\rangle$$

$$01 \longrightarrow |\psi^+\rangle$$

$$10 \longrightarrow |\psi^-\rangle$$

$$11 \longrightarrow |\phi^-\rangle$$

Quantum Channel



$$\rho_{00}$$

$$\rho_{01}$$

$$\rho_{10}$$

$$\rho_{11}$$

$$P_{error}(00) = 1 - \langle \phi^+ | \rho_{00} | \phi^+ \rangle$$

$$P_{error}(01) = 1 - \langle \psi^+ | \rho_{01} | \psi^+ \rangle$$

.....

.....

$$P_{error} = \frac{P_{error}(00) + P_{error}(01) + \dots}{4}$$



$$P_{error}(M) = 1 - Tr\left(E_M \Phi^{\otimes n}(\rho_M^{(n)})\right)$$

$$\bar{P}_{error} = \frac{\sum_M 1 - Tr\left(E_M \Phi^{\otimes n}(\rho_M^{(n)})\right)}{|M|}$$

Achievable rate for the source X

$$R(X) = \lim_{n \rightarrow \infty} \frac{\log |M|}{n} = \lim_{n \rightarrow \infty} \frac{k}{n}$$

when $P_{error} \rightarrow 0$

تعریف ظرفیت کلاسیک یک کانال کوانتومی

$$C := \underset{X}{\text{Max}} R(X)$$

ماکزیمم گیری روی تمام منابع ها صورت می گیرد.

این همان تعریف کلی و دقیق ظرفیت است و هنوز فرمول بسته نیست.

فرمول بسته برای ظرفیت کلاسیک کانال کوانتومی؟

فقط در یک حالت وجود دارد.

$C^{(1)}$

$\rho_1 \otimes \rho_2 \otimes \dots \otimes \rho_n$

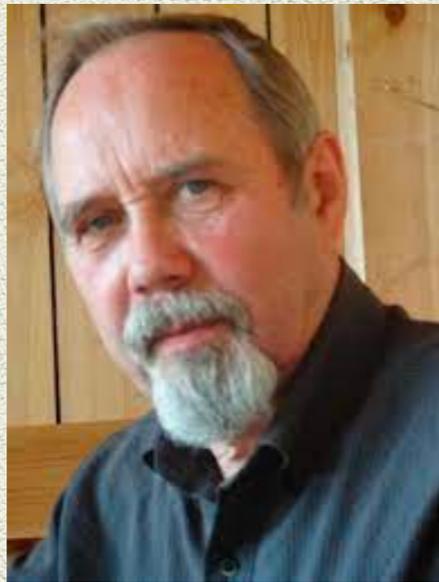


$\{E_M, \dots\}$

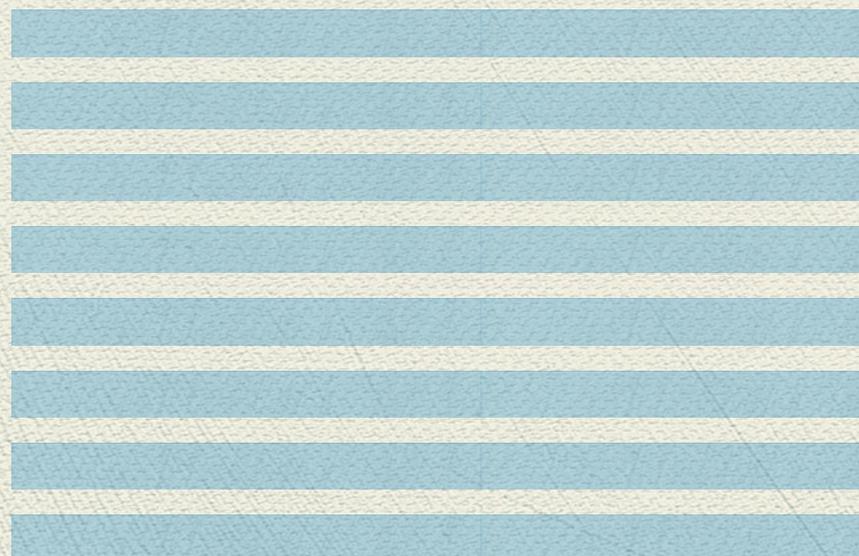
وقتی حالت های ورودی به صورت ضربی باشند: $\rho_1 \otimes \rho_2 \otimes \dots \otimes \rho_n$

فرم بسته برای ظرفیت کلاسیک کانال کوانتومی در یک حالت خاص: $C^{(1)}$

Holevo-Schumacher-Westmoreland



$$\rho_1 \otimes \rho_2 \otimes \dots \otimes \rho_M$$



$$\{E_M, \dots\}$$

Holevo-Schumacher-Westmoreland Or Product State Capacity

$$C^{(1)} = \chi^*(\Phi)$$

$$\chi^*(\Phi) = \underset{\{p_i, \rho_i\}}{\text{Max}} \chi(\{p_i, \rho_i\})$$

$$\chi(\Phi) = S\left[\sum_i p_i \Phi(\rho_i)\right] - \sum_i p_i S(\Phi(\rho_i))$$

Example 1: $C^{(1)}$ Capacity of **Depolarizing** Channel

$$\Phi(\rho) = (1 - p)\rho + \frac{p}{2}I$$

We take an ensemble $\{p_i, \rho_i\}$ and calculate

$$\chi(\Phi) = S\left[\sum_i p_i \Phi(\rho_i)\right] - \sum_i p_i S(\Phi(\rho_i))$$

$$\sum_i p_i \Phi(\rho_i) = \Phi\left(\sum_i p_i \rho_i\right) = \Phi(\rho) \qquad \Phi(\rho_i) = (1 - p)\rho_i + \frac{p}{2}I$$

$$\chi(\{p_i, \rho_i\}) = S\left[(1-p)\rho + p\frac{I}{2}\right] - \sum_i p_i S\left[(1-p)\rho_i + \frac{p}{2}I\right]$$

The input ensemble can be taken to be pure,
Thm. 13.3.2. Mark Wilde ([arxiv.org.1106.1445.](https://arxiv.org/abs/1106.1445)).

To maximize the first term, we take $\rho = \frac{I}{2}$.

For the second term:

$$\text{Since } S(U\sigma U^\dagger) = S(\sigma) \quad \forall U, \sigma$$

$$S\left[(1-p)|\psi_i\rangle\langle\psi_i| + p\frac{I}{2}\right] = S\left[(1-p|+\rangle\langle+| + p\frac{I}{2}\right]$$

**Therefore it is important to know
the minimum output entropy states.**

$$\chi(\{p_i, \rho_i\}) = S\left[(1-p)\rho + p\frac{I}{2}\right] - \sum_i p_i S\left[(1-p)\rho_i + \frac{p}{2}I\right]$$

$$= 1 - S\left[(1-p)\left|+\right\rangle\langle+| + \frac{p}{2}I\right]$$

$$C^{(1)}(p) = 1 + \frac{p}{2} \log \frac{p}{2} + \left(1 - \frac{p}{2}\right) \log \left(1 - \frac{p}{2}\right)$$

$$C^{(1)}(0) = 1$$

$$C^{(1)}(1) = 0$$

Example 2: $C^{(1)}$ Capacity of **Bit-flip** Channel

$$\Phi(\rho) = (1 - p)\rho + pX\rho X$$

We take $\rho = \frac{I}{2}$ to maximize the first term.

$$\chi(\{p_i, \rho_i\}) = S[(1 - p)\rho + pX\rho X] - \sum_i p_i S[(1 - p)\rho_i + pX\rho_i X]$$

Intuition: $\{p_i, |\psi_i\rangle\} = \{1/2, |+\rangle; 1/2, |-\rangle\}$


Invariant States

$$\chi(\{p_i, \rho_i\}) = S[(1-p)\rho + pX\rho X] - \sum_i p_i S[(1-p)\rho_i + pX\rho_i X]$$

Intuition: $\{p_i, |\psi_i\rangle\} = \{1/2, |+\rangle; 1/2, |-\rangle\}$

$$\chi = S\left(\frac{I}{2}\right) - \frac{1}{2}S(|+\rangle\langle +|) - \frac{1}{2}S(|-\rangle\langle -|) = 1$$

C=1 as it should be.

Example 3: $C^{(1)}$ Capacity of **Amplitude-Damping** Channel

$$\Phi(\rho) = A_0\rho A_0^\dagger + A_1\rho A_1^\dagger$$

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} \quad A_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$$

Intuition: $\{p_i, |\psi_i\rangle\} = \{p, |0\rangle; 1-p, |\psi\rangle\}$

Intuition: $\{p_i, |\psi_i\rangle\} = \{p, |0\rangle; 1 - p, |\psi\rangle\}$

$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$|0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$|\psi\rangle\langle\psi| = \begin{pmatrix} |a|^2 & a\bar{b} \\ \bar{a}b & |b|^2 \end{pmatrix}$$

$$\rho = \begin{pmatrix} p + (1 - p)|a|^2 & (1 - p)a\bar{b} \\ (1 - p)\bar{a}b & (1 - p)|b|^2 \end{pmatrix}$$

$$\Phi(|0\rangle\langle 0|) = |0\rangle\langle 0|$$

$$\Phi(|\psi\rangle\langle\psi|) = \begin{pmatrix} |a|^2 + \gamma|b|^2 & a\bar{b} \\ \bar{a}b & (1 - \gamma)|b|^2 \end{pmatrix}$$

$$\Phi(\rho) = \begin{pmatrix} p + (1 - p)|a|^2 + \gamma(1 - p)|b|^2 & (1 - p)a\bar{b} \\ (1 - p)\bar{a}b & (1 - \gamma)(1 - p)|b|^2 \end{pmatrix}$$

$$\chi_\gamma(p, a) = S[\Phi(\rho)] - (1 - p)S[\Phi(|\psi\rangle\langle\psi|)]$$

$$\chi(p, a) = S[\Phi(\rho)] - (1 - p)S[\Phi(|\psi\rangle\langle\psi|)]$$

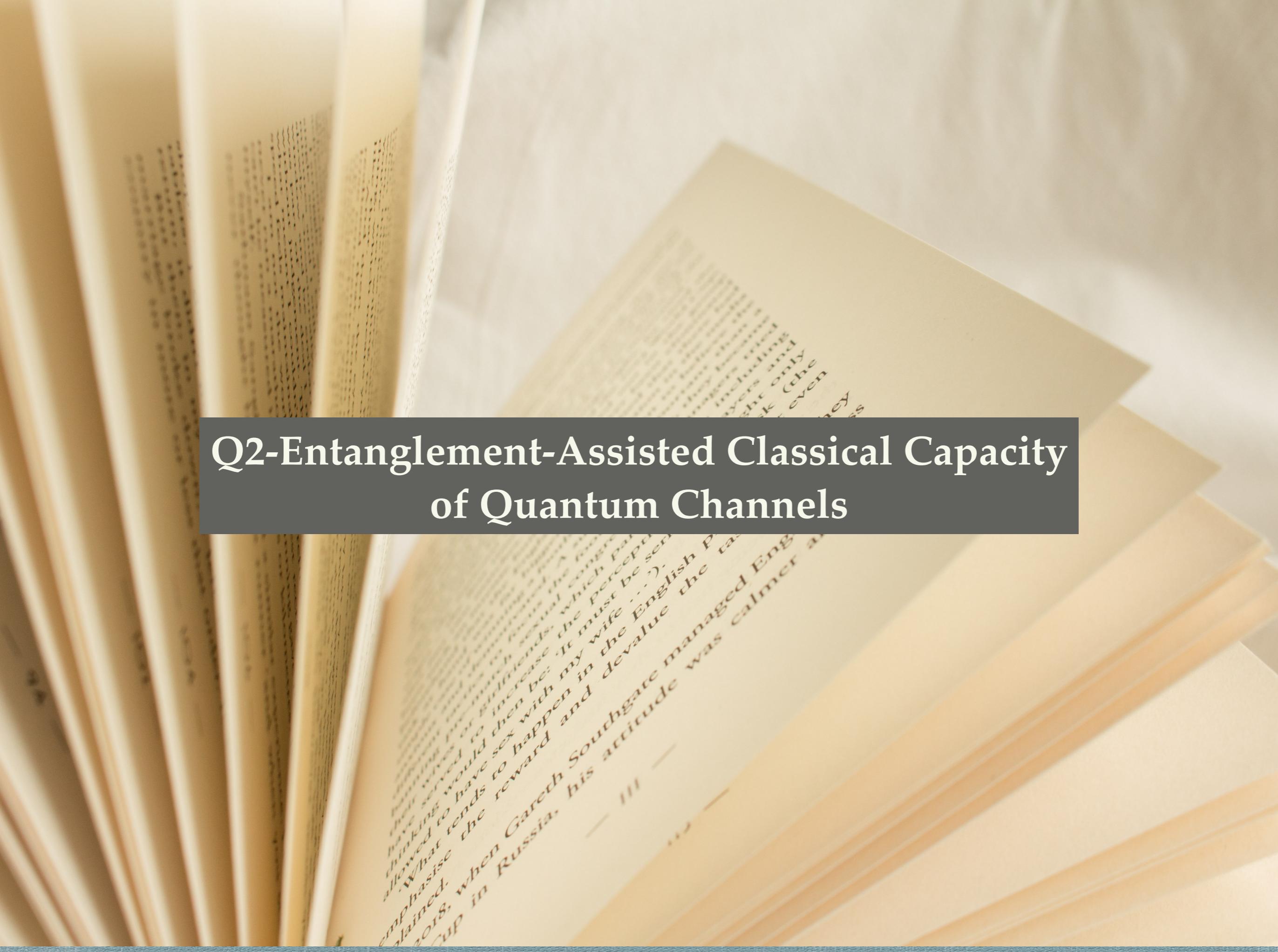
Using $S(\sigma) = S(U\sigma U^\dagger)$ we can take a and b to be real.

$$a = \cos \theta$$

$$b = \sin \theta$$

$$\chi(p, \theta) = S[\Phi(\rho)] - (1 - p)S[\Phi(|\psi\rangle\langle\psi|)]$$

χ^* and hence C can be calculated only numerically.



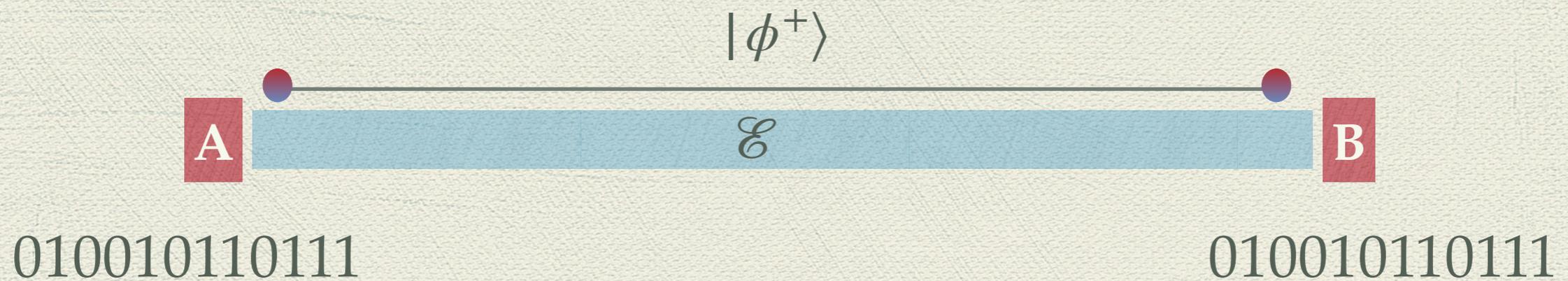
Q2-Entanglement-Assisted Classical Capacity of Quantum Channels

... have served to increase the number of children, which has allowed to have sex with my wife ...
emphasise the reward and devalue the ta...
2018, when Gareth Southgate managed Eng...
Cup in Russia, his attitude was calmer a...

Q2: Entanglement-Assisted Classical Capacity of Quantum Channels

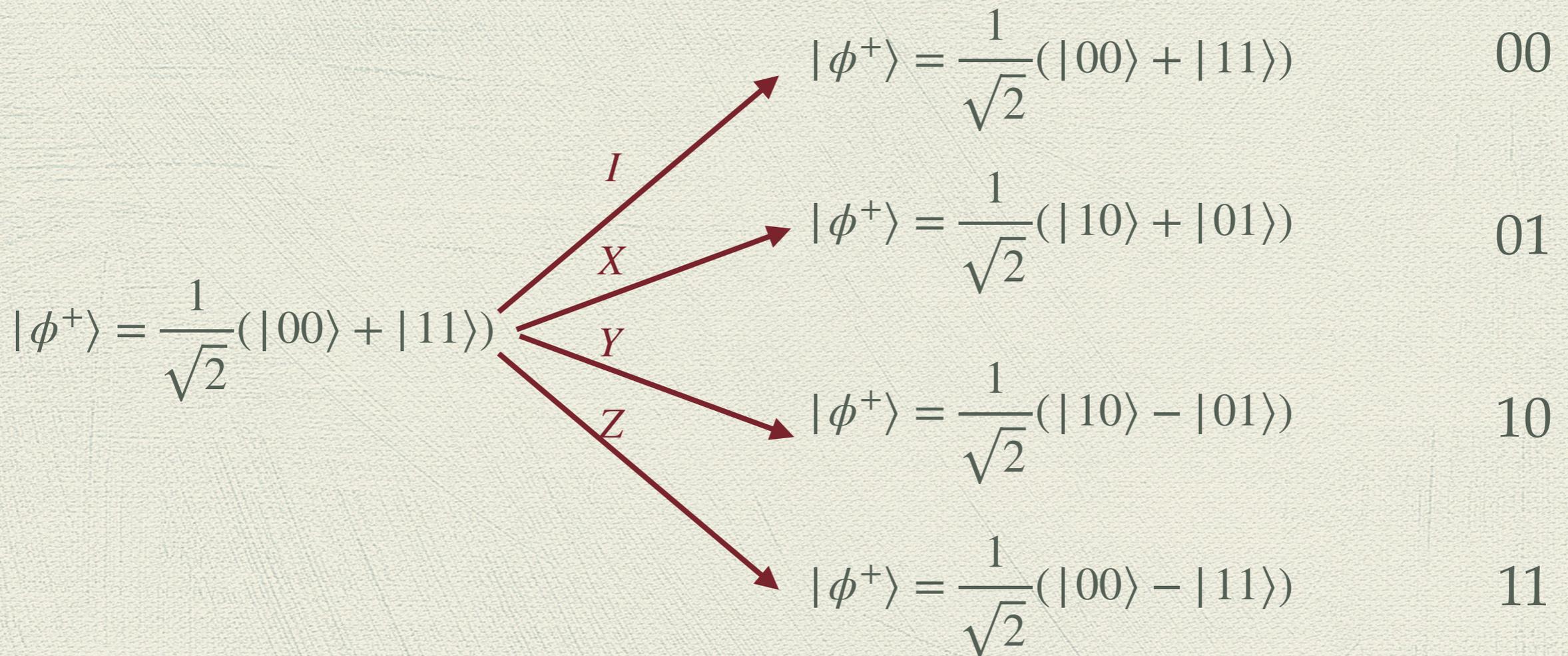
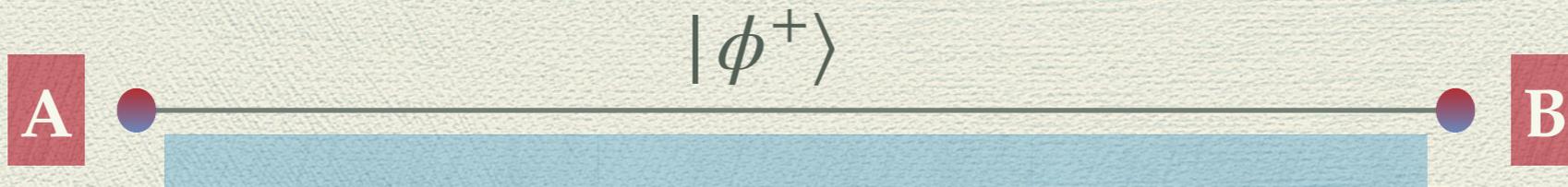
بین فرستنده و گیرنده به دلخواه حالت های درهم تنیده ماکزیمال وجود دارد که می توانند از آن برای ارسال پیام های کلاسیک استفاده کنند.

Definition: Entanglement-assisted Classical Capacity
of Quantum Channels

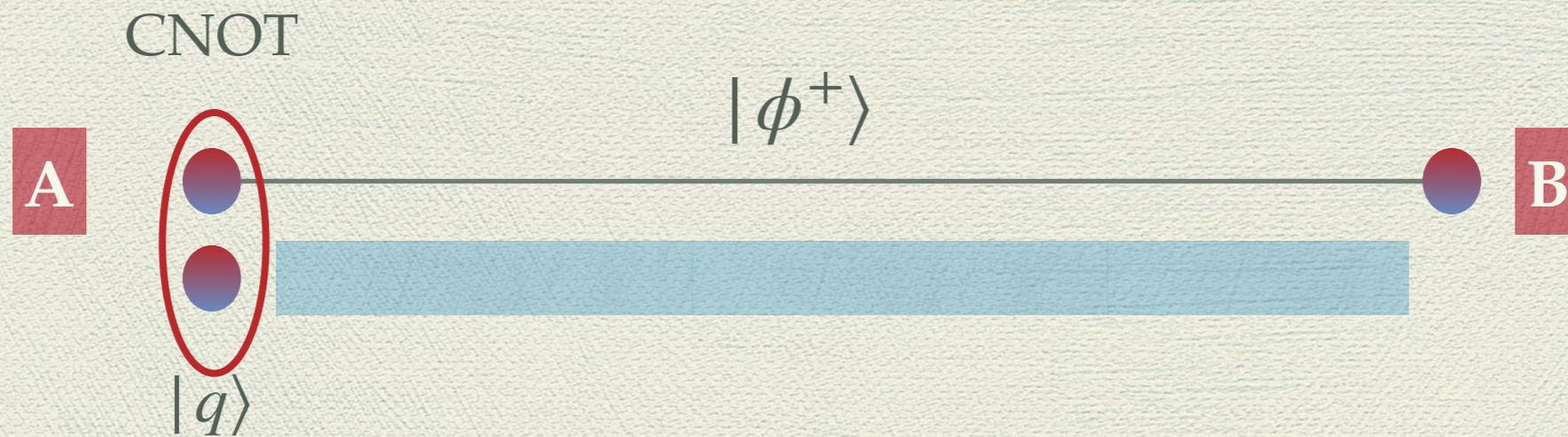


Example: Dense Coding

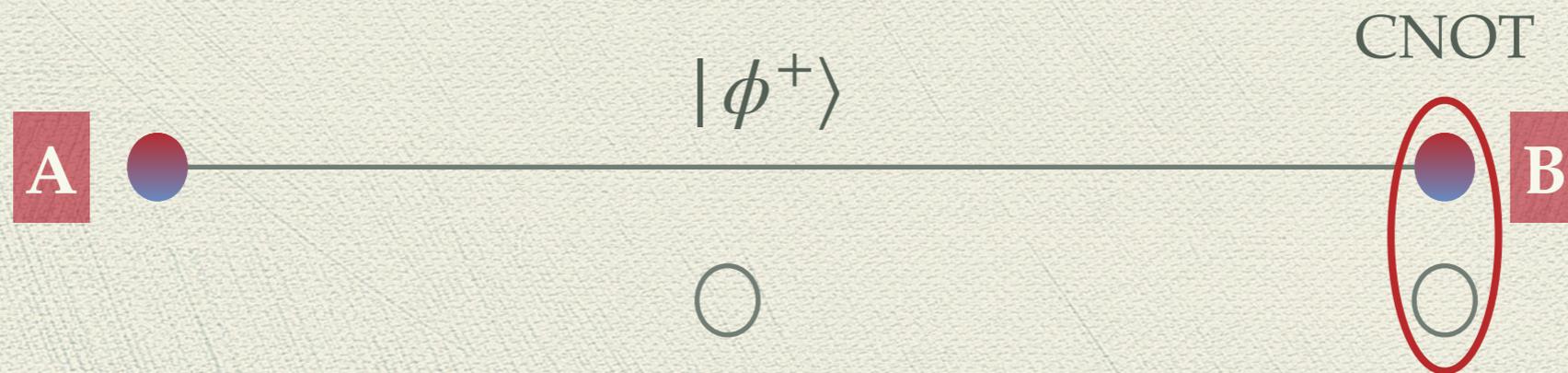
Example 1: Dense Coding



Example 2



$$|\Psi\rangle_{ABA} = \frac{1}{\sqrt{2}}(|00q\rangle + |11\bar{q}\rangle)$$



General Definition: Entanglement-assisted Classical Capacity of Quantum Channels

$$|\phi^+\rangle^{\otimes \infty}$$



$\rho_M^{(n)}$ = Quantum state of n-qubits

$$C = \text{Max}_{n \rightarrow \infty} \frac{|M|}{n} \quad ?$$

Calculation: Entanglement-Assisted Classical Capacity of Quantum Channel

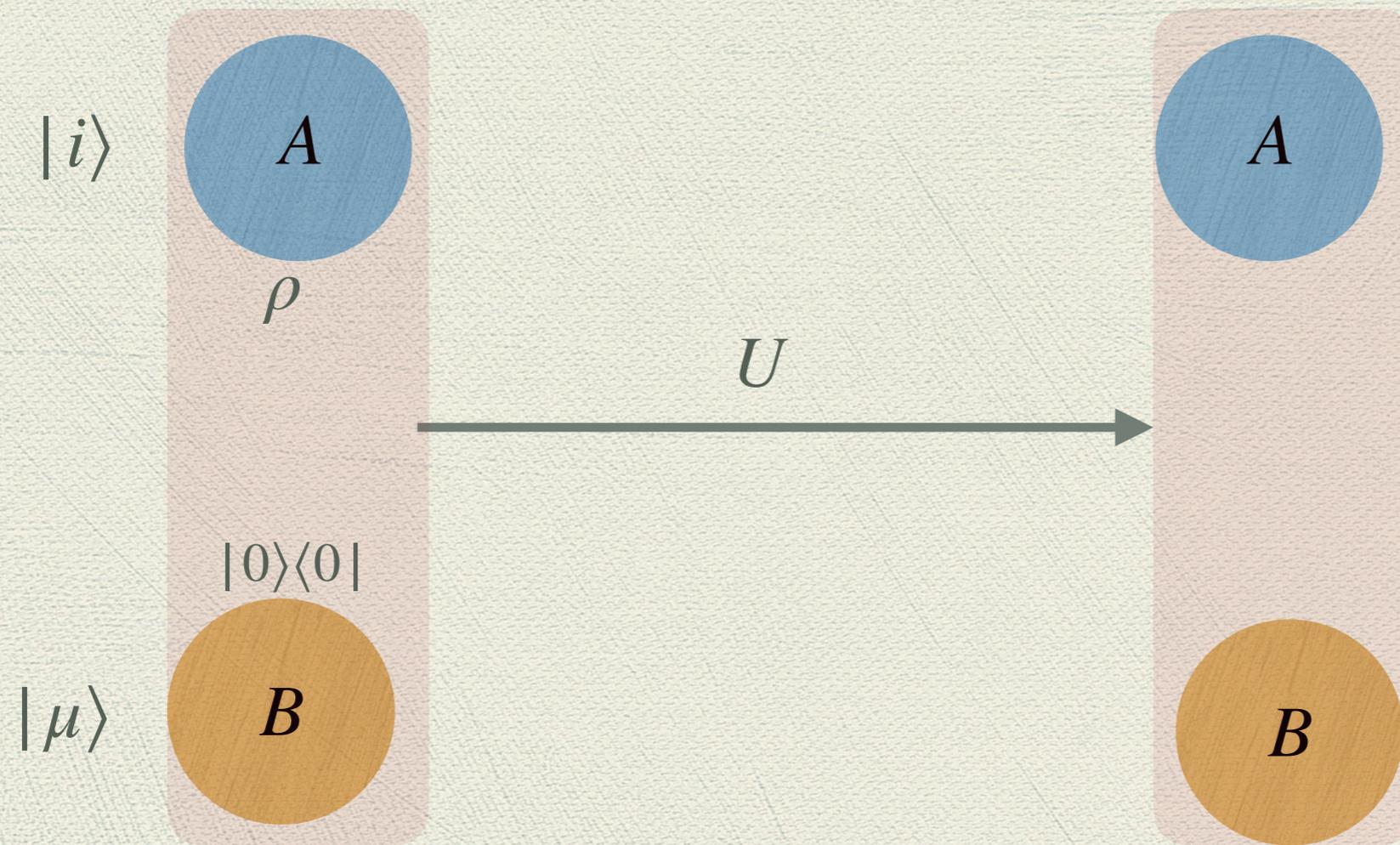
$$C_E(\Phi) = \underset{\rho}{\text{Max}} I(\rho, \Phi)$$

$$I(\rho, \Phi) = S(\rho) + S(\Phi(\rho)) - S(\Phi^c(\rho))$$

بنابراین برای این نوع ظرفیت یک فرمول بسته و ساده وجود دارد.

ولی بازهم می بایست این عبارت آخری را روی تمام ماتریس های چگالی بیشینه کرد.

Complementary Channels

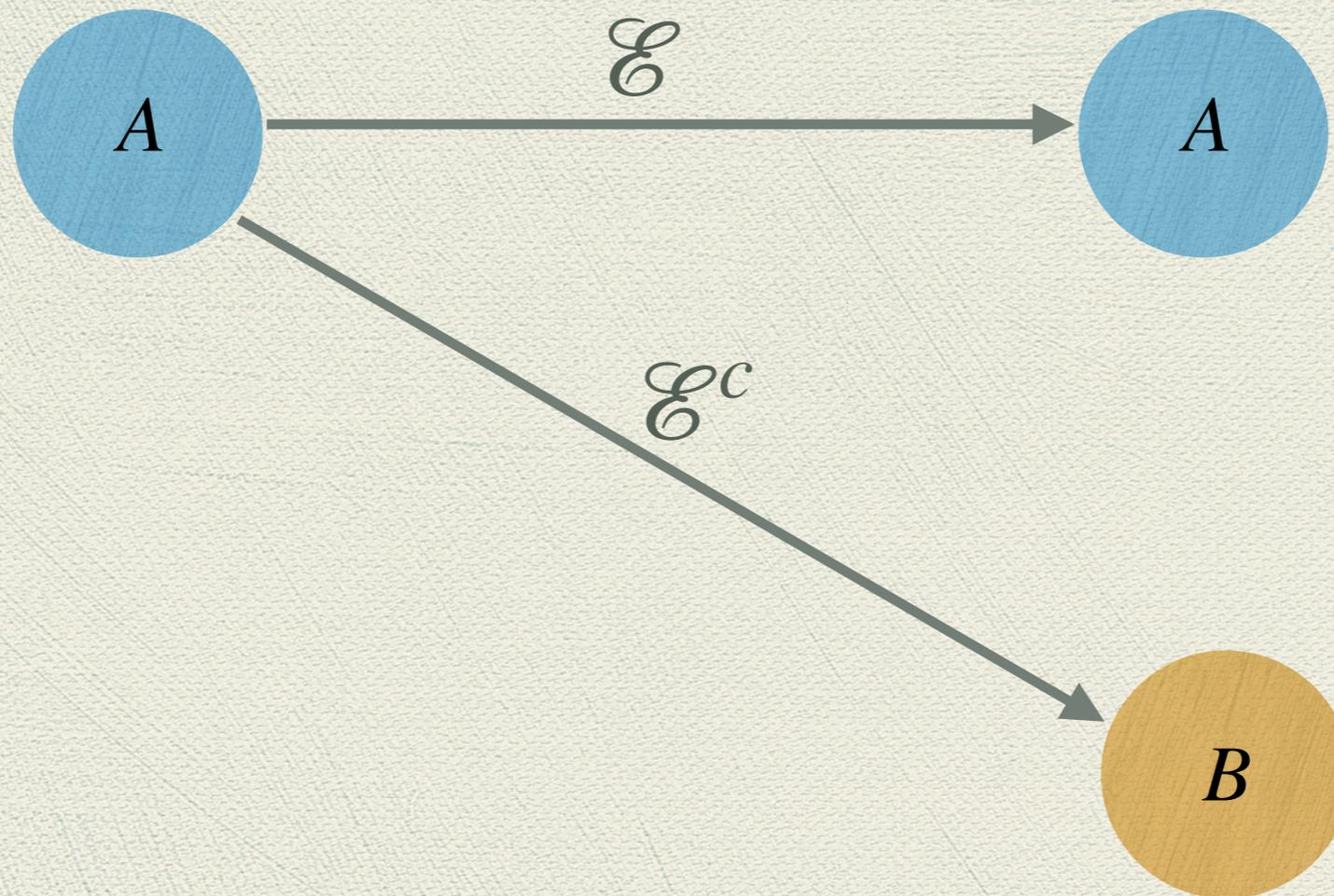


Complementary Channels

$$\mathcal{E}(\rho) = \text{Tr}_B \left[U(\rho \otimes |0\rangle\langle 0|)U^\dagger \right] \in A$$

$$\mathcal{E}^c(\rho) = \text{Tr}_A \left[U(\rho \otimes |0\rangle\langle 0|)U^\dagger \right] \in B$$

Complementary Channels



Kraus Decomposition

$$\mathcal{E}(\rho) = \sum_{\mu} A_{\mu} \rho A_{\mu}^{\dagger}$$

$$(K_i)_{\mu,j} = (A_{\mu})_{ij}$$

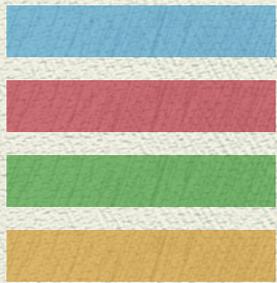
$$\mathcal{E}^c(\rho) = \sum_i K_i \rho K_i^{\dagger}$$

ماتریس K_1 را از سطرهای اول A ها می سازیم.

ماتریس K_2 را از سطرهای دوم A ها می سازیم.

ماتریس K_3 را از سطرهای سوم A ها می سازیم.

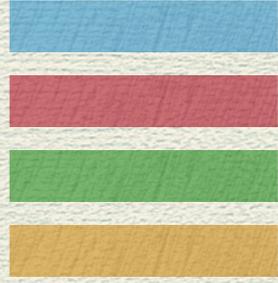
\mathcal{E}



A_1



A_2



A_3

\mathcal{E}^c



K_1



K_2



K_3



K_4

Example 1: Complement of Bit-flip Channel

$$\mathcal{E}(\rho) = (1 - p)\rho + p\sigma_x\rho\sigma_x$$

$$A_0 = \begin{pmatrix} \sqrt{1-p} & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} \quad A_1 = \begin{pmatrix} 0 & \sqrt{p} \\ \sqrt{p} & 0 \end{pmatrix}$$

$$K_0 = \begin{pmatrix} \sqrt{1-p} & 0 \\ 0 & \sqrt{p} \end{pmatrix} \quad K_1 = \begin{pmatrix} 0 & \sqrt{1-p} \\ \sqrt{p} & 0 \end{pmatrix}$$

$$\rho = \frac{1}{2}(I + \mathbf{r} \cdot \sigma)$$

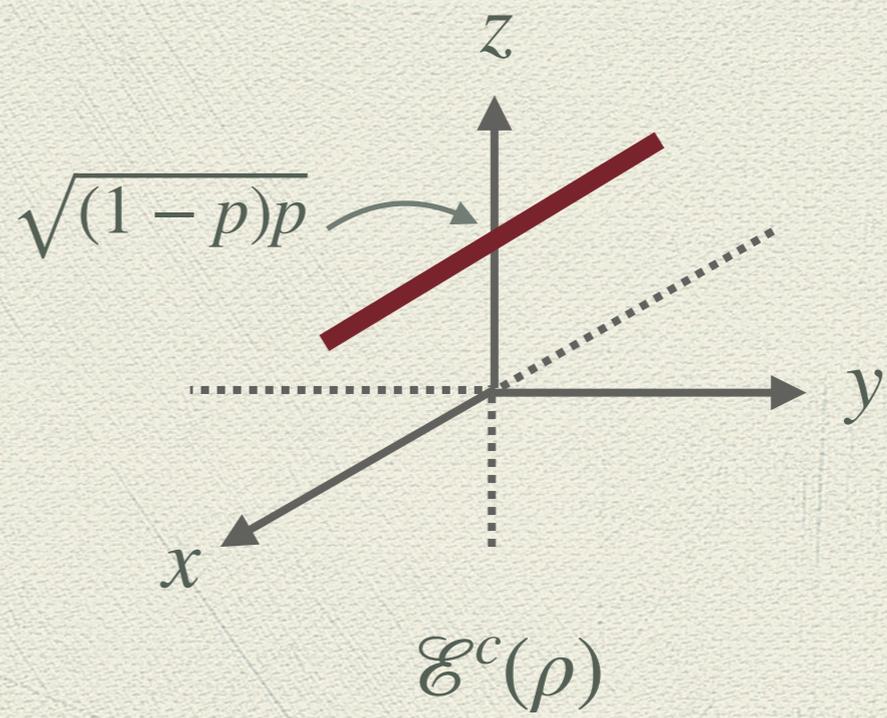
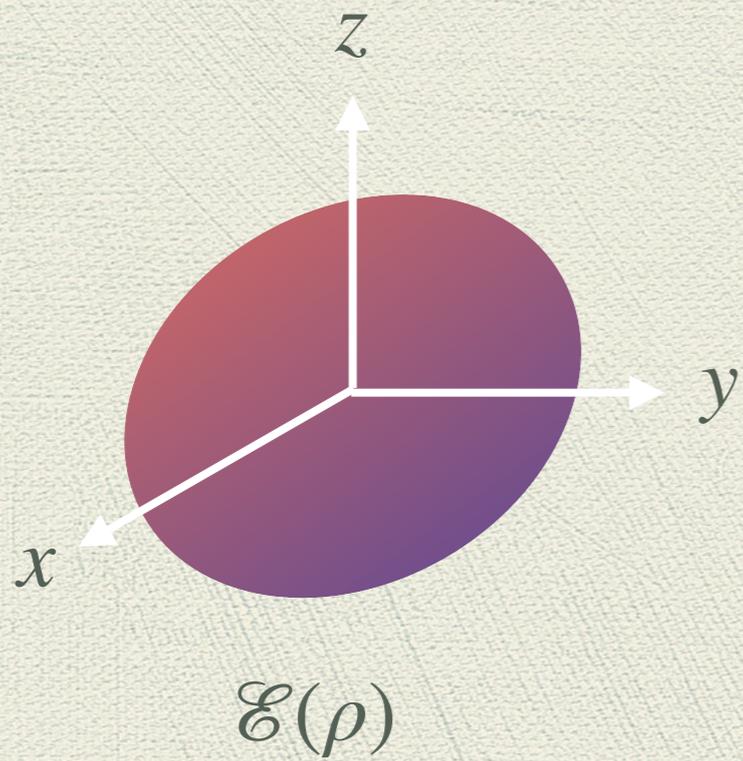
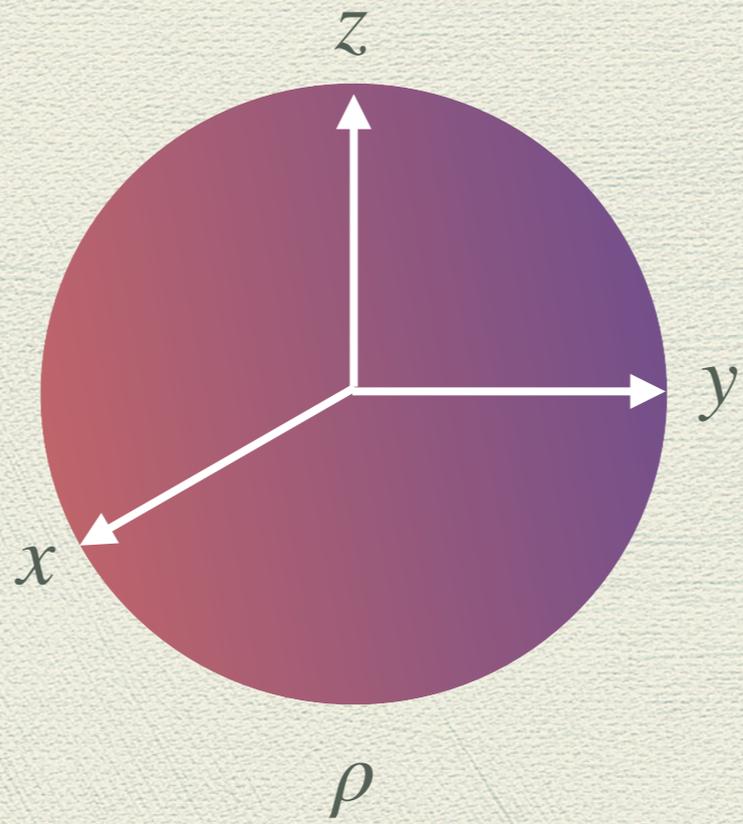
$$\mathbf{r} = (x, y, z)$$

$$\mathcal{E}(\rho) = \frac{1}{2}(I + \mathbf{r}' \cdot \sigma)$$

$$\mathbf{r}' = (x, (1 - 2p)y, (1 - 2p)z)$$

$$\mathcal{E}^c(\rho) = \frac{1}{2}(I + \mathbf{r}_c' \cdot \sigma)$$

$$\mathbf{r}_c' = (\sqrt{(1-p)p} x, 0, (1 - 2p))$$



$$I(\rho, \Phi) = S(\rho) + S(\Phi(\rho)) - S(\Phi^c(\rho))$$

Example 2: Depolarizing Channel

$$\mathcal{E}(\rho) = \left(1 - \frac{3p}{4}\right)\rho + \frac{p}{4}(\sigma_x\rho\sigma_x + \sigma_y\rho\sigma_y + \sigma_z\rho\sigma_z)$$

$$A_0 = \alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A_1 = \beta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A_2 = \beta \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$A_3 = \beta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\alpha = \sqrt{1 - \frac{3p}{4}}$$

$$\beta = \sqrt{\frac{p}{4}}$$

$$A_0 = \alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A_1 = \beta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A_2 = \beta \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$A_3 = \beta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$R_1 = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \\ 0 & -i\beta \\ \beta & 0 \end{pmatrix}$$

$$R_2 = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \\ i\beta & 0 \\ 0 & -\beta \end{pmatrix}$$

$$\mathcal{E}^c(\rho) = \begin{pmatrix} \alpha^2 & \alpha\beta x & \alpha\beta y & \alpha\beta z \\ \alpha\beta x & \beta & -i\beta z & i\beta y \\ \alpha\beta y & i\beta z & \beta & -i\beta x \\ \alpha\beta z & -i\beta y & i\beta x & \beta \end{pmatrix}$$

Eigenvalues of $\rho : \{\frac{1}{2}(1 \pm r)\}$

Eigenvalues of $\mathcal{E}(\rho) : \{\frac{1}{2}(1 \pm (1 - p)r)\}$

Eigenvalues of $\mathcal{E}^c(\rho) : \{\beta^2(1 \pm r), \frac{1}{2}(1 - 2\beta^2 \pm \sqrt{(1 - 4\beta^2)^2 - 8(\beta^2 r^2)^2})\}$

$$I(\rho, \Phi) = S(\rho) + S(\Phi(\rho)) - S(\Phi^c(\rho))$$

Q3-Quantum Capacity of Quantum Channels

... have served to increase the ...
... thinking would then be: 'It must ...
... What tends to happen in the Em ...
... emphasise the reward and devalue ...
... 2018, when Gareth Southgate manage ...
... Cup in Russia, his attitude was cal ...

... between players and ...
... of the task (the ...
... us if I'm not even ...
... ess is that they ...
... Harkness ...
... and at the ...
... d more

Definition: Quantum Capacity
of Quantum Channels



$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Quantum Channel Φ



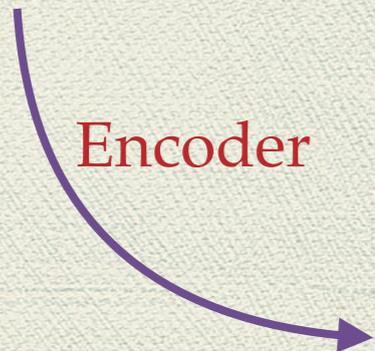
ρ

Definition: Quantum Capacity of Quantum Channels

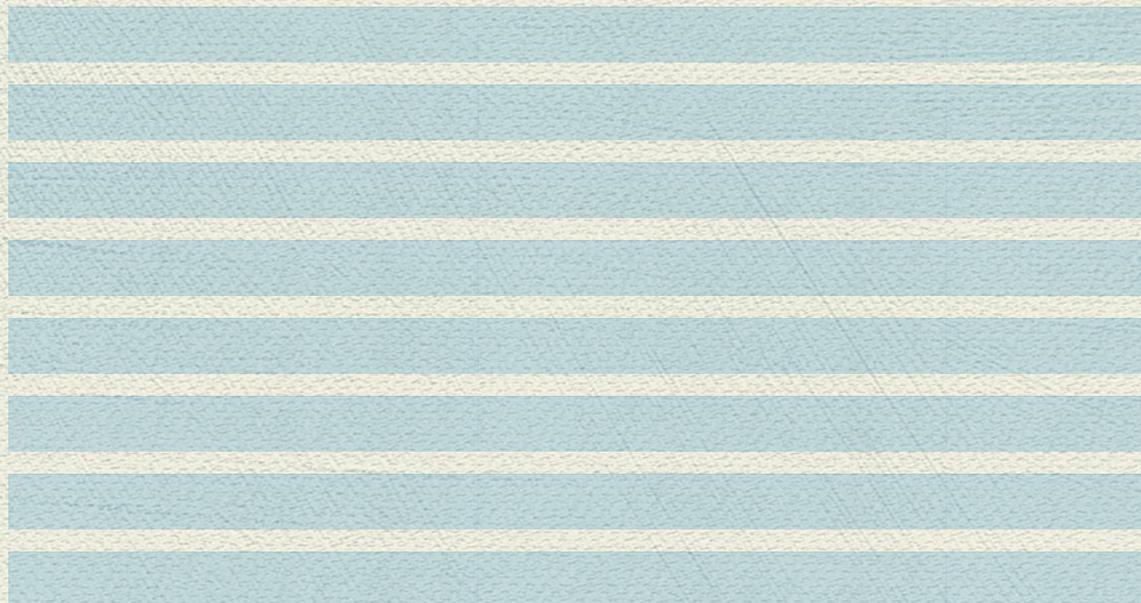
$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$



Encoder

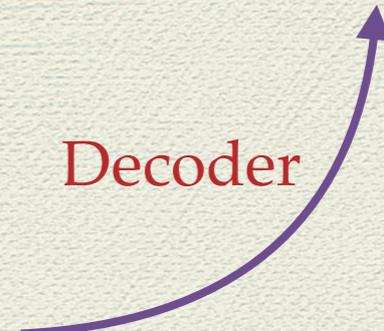


$$|\Psi\rangle_n$$



$$\Phi^{\otimes n}(|\Psi\rangle_n)$$

Decoder



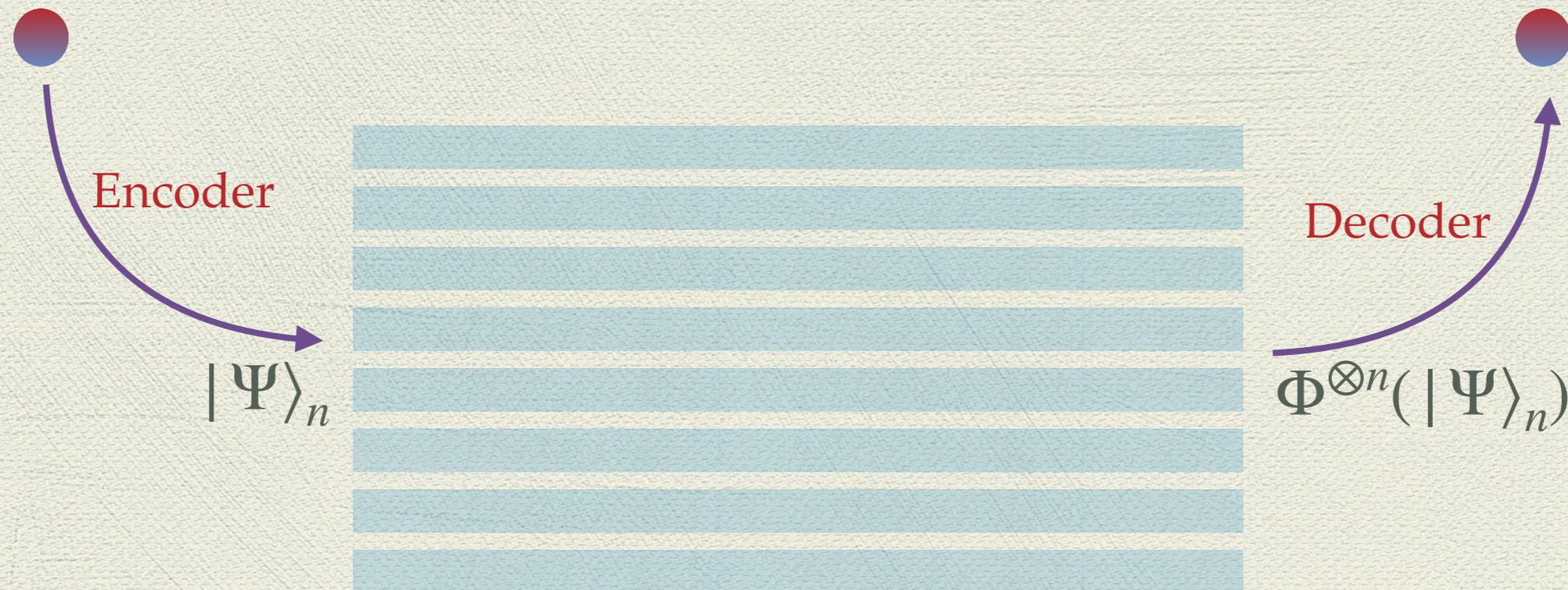
$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$



Definition: Quantum Capacity of Quantum Channels

$$X = \{p_i, |\psi_i\rangle \in C^k\}$$

$$Y = \{p_i, |\psi'_i\rangle \in C^k\}$$



We want $\bar{F}(|\psi'_k\rangle, |\psi_k\rangle) \geq 1 - \epsilon$

$$R(X) = \lim_{n \rightarrow \infty} \frac{k}{n}$$

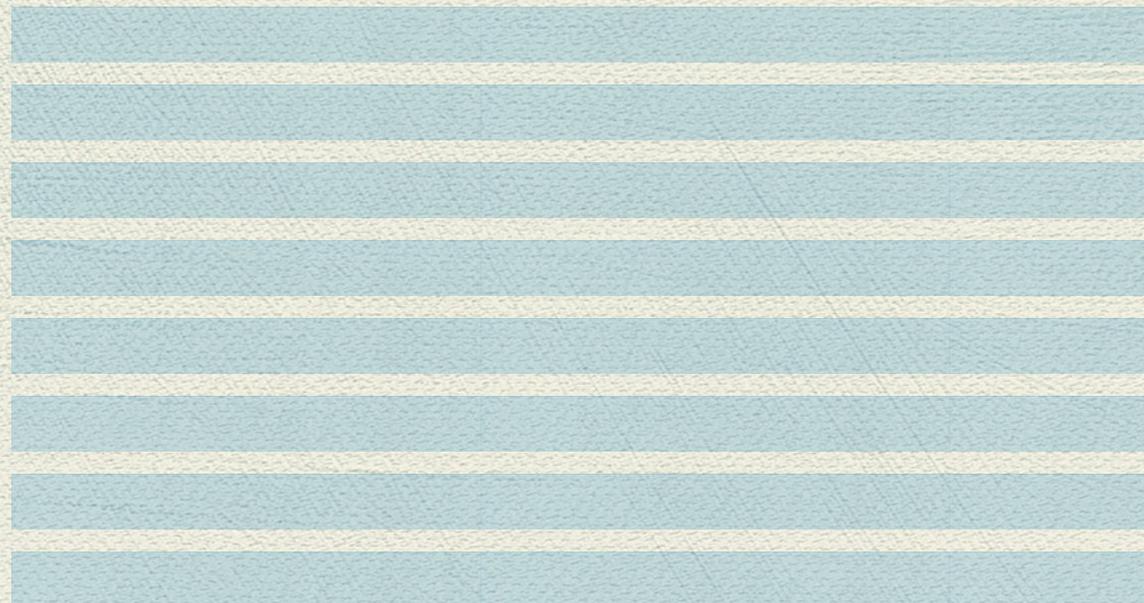
Definition: Quantum Capacity of Quantum Channels

$$X = \{p_i, |\psi_i\rangle \in C^k\}$$



Encoder

$$|\Psi\rangle_n$$



$$Y = \{p_i, |\psi'_i\rangle \in C^k\}$$



Decoder

$$\Phi^{\otimes n}(|\Psi\rangle_n)$$

$$C = \underset{X}{\text{Max}} R(X)$$

Calculation: Quantum Capacity of Quantum Channels

$$J_n(\rho, \Phi) = S(\Phi^{\otimes n}(\rho)) - S((\Phi^c)^{\otimes n}(\rho))$$

$$J_n(\Phi) = \underset{\rho}{\text{Max}} J_n(\rho, \Phi^{\otimes n})$$

$$C = \lim_{n \rightarrow \infty} \frac{1}{n} J_n[\Phi]$$

Obviously this capacity cannot be calculated,

Except when the channel is degradable.

For degradable channels

$$C = J[\Phi] = \underset{\rho}{\text{Max}} S[\Phi(\rho)] - S[\Phi^c(\rho)]$$

Q4-Private Capacity of Quantum Channels

... have served to increase the ...
... thinking would then be: 'It must ...
... What tends to happen in the Em ...
... emphasise the reward and devalue ...
... 2018, when Gareth Southgate manage ...
... Cup in Russia, his attitude was cal ...

... between players and ...
... of the task (the ...
... us if I'm not even ...
... ess is that they ...
... Harkness ...
... and at the ...
... d more

M=Classical Message

M'



$$Pr(M \neq M') \leq \epsilon$$

$$\|\rho_{BE} - \rho_B \otimes \rho_E\| \leq \delta$$

Transferring classical messages to Bob without any leakage to Eve

Single Shot Capacity

$$C_p^1(\Lambda) = \max_{\{p_i, \rho_i\}} [\chi(\Lambda) - \chi(\Lambda^c)]$$

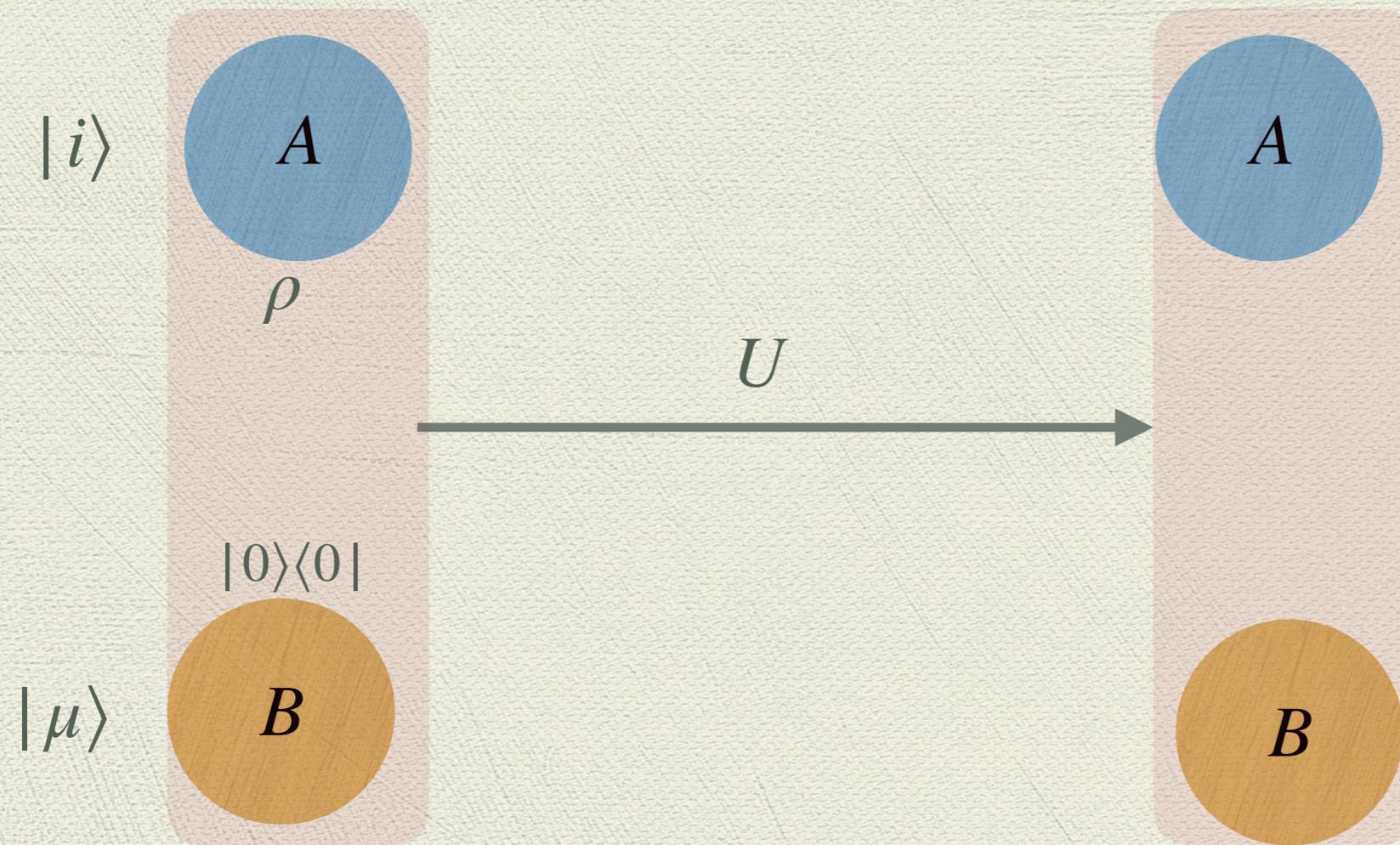
$$C_p(\Lambda) = \max_{\{p_i, \rho_i\}} \lim_{n \rightarrow \infty} [\chi(\Lambda^{\otimes n}) - \chi(\Lambda^{c \otimes n})]$$

$$C_p^1(\Lambda) \leq C_p(\Lambda)$$

$$Q(\Lambda) \leq C_p(\Lambda)$$

Degradable Channels

Complementary Channels

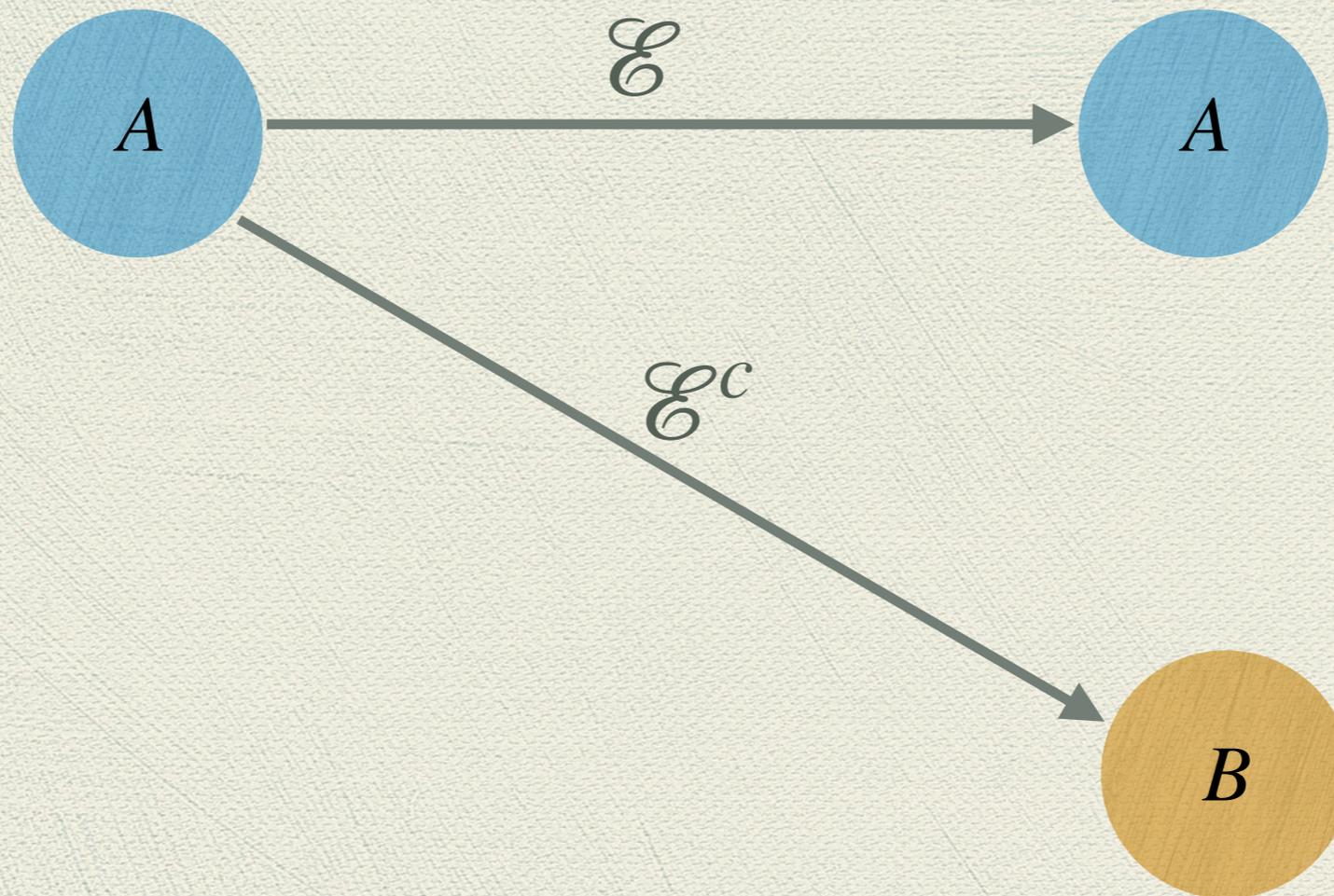


Complementary Channels

$$\mathcal{E}(\rho) = \text{Tr}_B \left[U(\rho \otimes |0\rangle\langle 0|)U^\dagger \right] \in A$$

$$\mathcal{E}^c(\rho) = \text{Tr}_A \left[U(\rho \otimes |0\rangle\langle 0|)U^\dagger \right] \in B$$

Complementary Channels



Kraus Decomposition

$$\mathcal{E}(\rho) = \sum_{\mu} A_{\mu} \rho A_{\mu}^{\dagger} \quad (A_{\mu})_{ij} = \langle i, \mu | U | j, 0 \rangle$$

$$\mathcal{E}^c(\rho) = \sum_i K_i \rho K_i^{\dagger} \quad (K_i)_{\mu,j} = \langle i, \mu | U | j, 0 \rangle$$

$$(K_i)_{\mu,j} = (A_{\mu})_{ij}$$

$$(K_i)_{\mu,j} = (A_\mu)_{ij}$$

d_A

d_A

d_B

K_i

d_A

d_A

A_μ

Example 1: Complement of Bit-flip Channel

$$(K_i)_{\mu,j} = (A_\mu)_{i,j}$$

$$\mathcal{E}(\rho) = (1-p)\rho + p\sigma_x\rho\sigma_x$$

$$A_0 = \begin{pmatrix} \sqrt{1-p} & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} \quad A_1 = \begin{pmatrix} 0 & \sqrt{p} \\ \sqrt{p} & 0 \end{pmatrix}$$

$$K_0 = \begin{pmatrix} \sqrt{1-p} & 0 \\ 0 & \sqrt{p} \end{pmatrix} \quad K_1 = \begin{pmatrix} 0 & \sqrt{1-p} \\ \sqrt{p} & 0 \end{pmatrix}$$

Example 2: Complement of Pauli Channel

$$\mathcal{E}(\rho) = p_0\rho + p_1\sigma_x\rho\sigma_x + p_2\sigma_y\rho\sigma_y + p_3\sigma_z\rho\sigma_z$$

$$K_0 = \begin{pmatrix} \sqrt{p_0} & 0 \\ 0 & \sqrt{p_1} \\ 0 & -i\sqrt{p_2} \\ \sqrt{p_3} & 0 \end{pmatrix} \quad K_1 = \begin{pmatrix} 0 & \sqrt{p_0} \\ \sqrt{p_1} & 0 \\ i\sqrt{p_2} & 0 \\ 0 & -\sqrt{p_3} \end{pmatrix}$$

More Examples

$$\mathcal{E}_0(\rho) = \text{Tr}(\rho) \frac{I}{d}$$

$$\tilde{\mathcal{E}}_0(\rho) = \frac{I}{d} (\rho \otimes I)$$

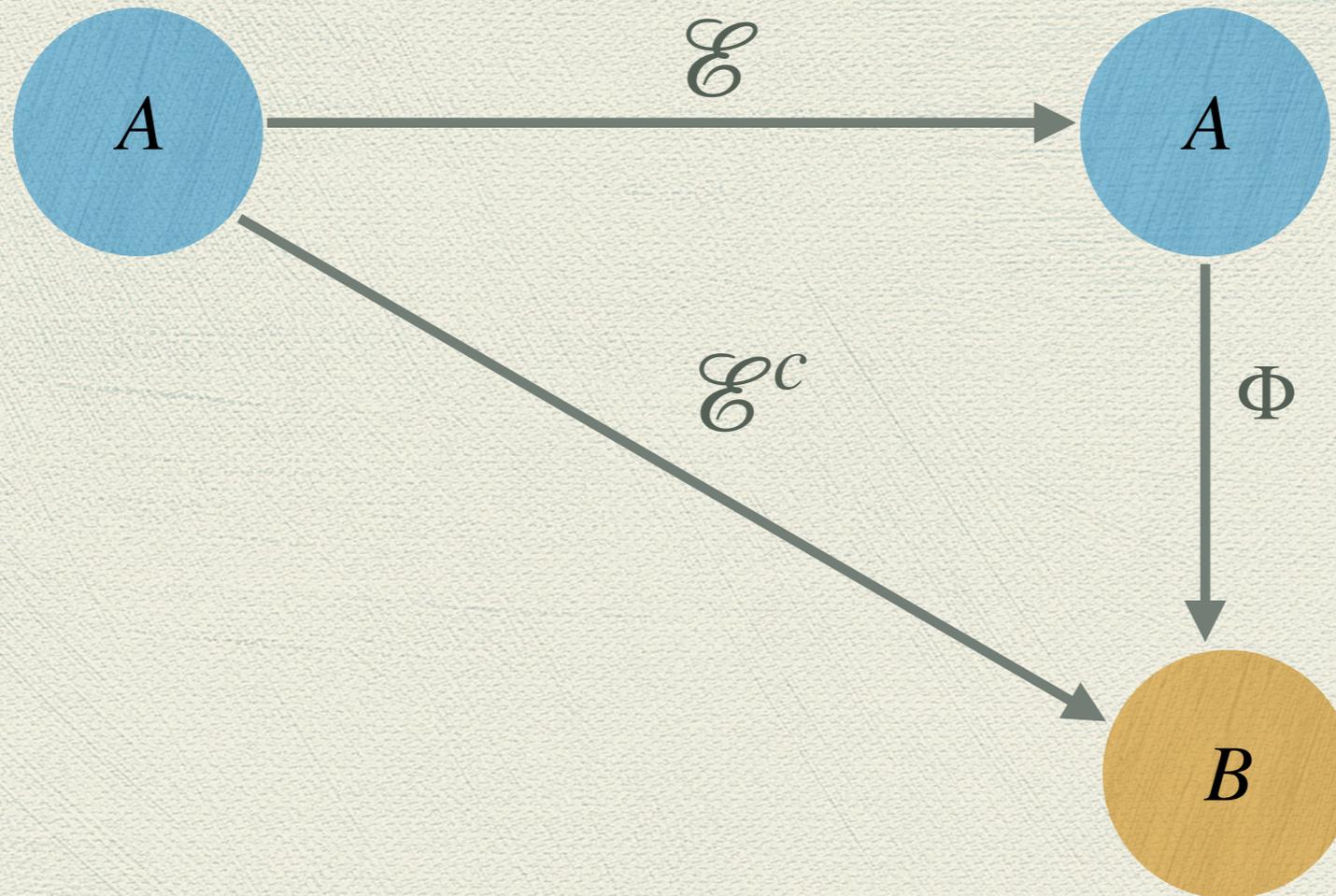
$$\mathcal{E}_U(\rho) = U\rho U^\dagger$$

$$\tilde{\mathcal{E}}_U(\rho) = \text{Tr}(\rho)$$

$$\mathcal{E}_\psi(\rho) = \text{Tr}(\rho) |\psi\rangle\langle\psi|$$

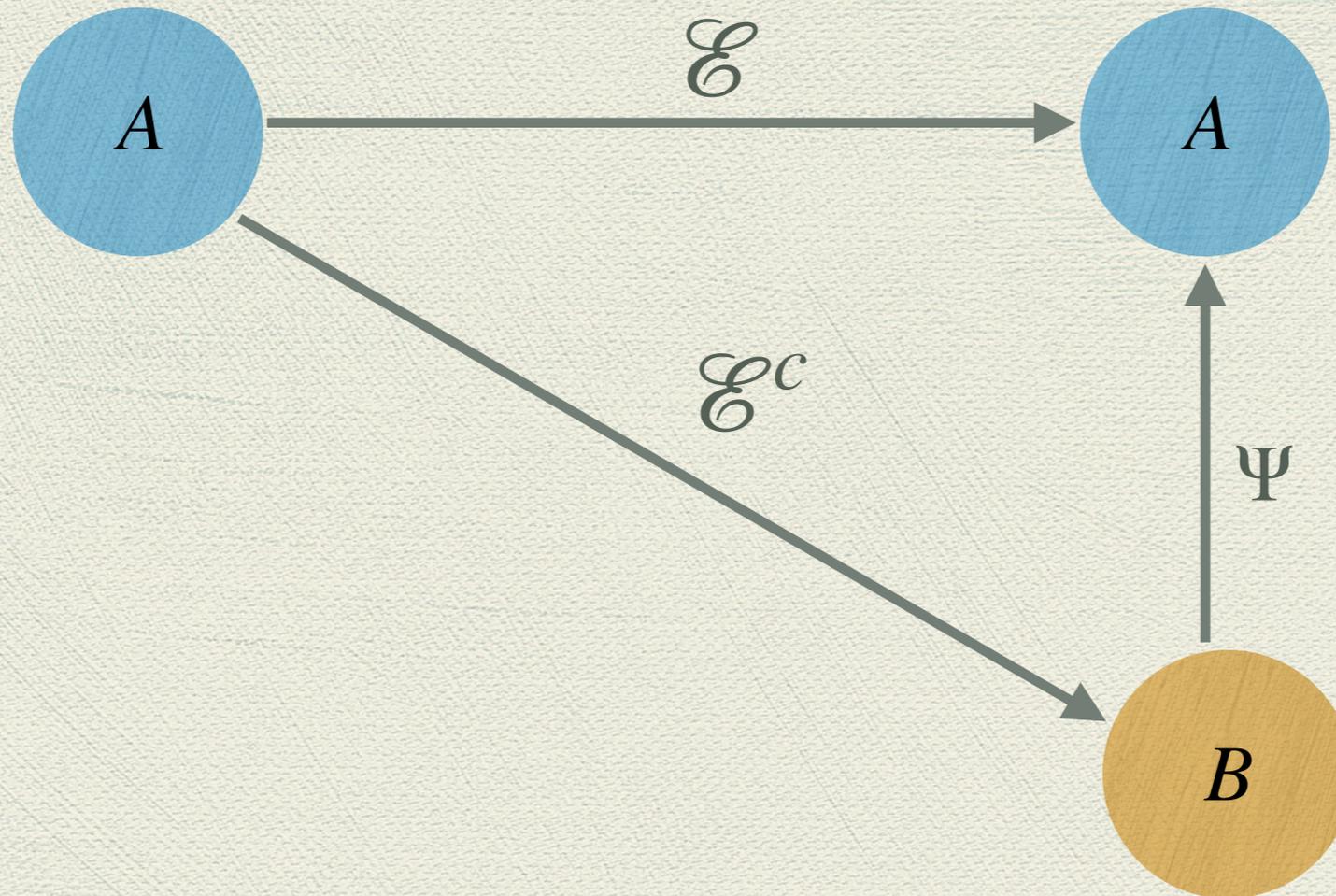
$$\mathcal{E}_\psi^c(\rho) = \rho$$

Degradable Channels



$$\mathcal{E}^c = \Phi \circ \mathcal{E}$$

Anti-Degradable Channels



$$\mathcal{E} = \Psi \circ \mathcal{E}^c$$

Examples of degradable and anti-degradable Channels

$$\mathcal{E}_0(\rho) = \text{Tr}(\rho) \frac{I}{d}$$

$$\tilde{\mathcal{E}}_0(\rho) = \frac{I}{d}(\rho \otimes I)$$

$$\Phi \circ \tilde{\mathcal{E}}_0 = \mathcal{E}_0$$

$$\Phi(X) = \text{Tr}_A(X)$$

Anti-Degradable

Examples of degradable and anti-degradable Channels

$$\mathcal{E}_U(\rho) = U\rho U^\dagger$$

$$\tilde{\mathcal{E}}_U(\rho) = \text{Tr}(\rho)$$

$$\tilde{\mathcal{E}}_U = \Phi \circ \mathcal{E}_U$$

$$\Phi(X) = \text{Tr}(X)$$

Degradable

Examples of degradable and anti-degradable Channels

$$\mathcal{E}_\psi(\rho) = \text{Tr}(\rho) |\psi\rangle\langle\psi| \quad \tilde{\mathcal{E}}_\psi(\rho) = \rho$$

$$\mathcal{E}_\psi = \Phi \circ \tilde{\mathcal{E}}_\psi$$

$$\Phi(X) = \text{Tr}(X) |\psi\rangle\langle\psi|$$

Anti-Degradable

Is there any non-trivial example of a degradable channel?

For which we can calculate the quantum capacity?

Q4: Private Capacity of Quantum Channels

For more information and for details, see:

[Capacities of the covariant Pauli channel](#)

A Poshtvan, V Karimipour

[Physical Review A 106 \(6\), 062408](#)

End of part III